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## Theorem 2

If $o(G) = p^n$ where $p$ is a prime number, then the centre $Z \neq \{e\}$

**Proof.** By the class equation of $G$, we have

$$o(G) = o(Z) + \sum_{a \notin Z} \frac{o(G)}{o(N(a))} \quad \text{--- (1)}$$

Where the summation runs over one element $a$ in each conjugate class containing more than one element.

Now $\forall\, a \in G$, $N(a)$ is a subgroup of $G$. Therefore, by Lagrange's theorem,

$o(N(a))$ is a divisor of $o(G)$. Also $a \in G \Rightarrow N(a) \neq G \Rightarrow o[N(a)] < o(G)$

Therefore if $a \notin Z$, then $o(N(a))$ must be of the form $p^{n_a}$ where $n_a$ is some integer such that $1 \leq n_a < n$. Suppose there are exactly $z$ elements in $Z$ i.e. Let $o(Z) = z$. Then the class equation (1) gives:

$$p^n = z + \sum \frac{p^n}{p^{n_a}}$$ where each $n_a$ is some integer such that $1 \leq n_a < n$.

$$\therefore Z = p^n - \sum \frac{p^n}{p^{n_a}} \quad \text{--- (2)}$$

Where $n_a$'s are some positive integers each being less than $n$.